M E T U Department of Mathematics

Math 261	Linear Algebra I	Exam 2	21.12.2008

Last Name: Name : Student No:	Instructor : *Mustafa Korkmaz* Time : 11:00 Duration : 110 *minutes*	Signature

5 QUESTIONS						TOTAL 90 POINTS	
1	2	3	4	5	SOLUTIONS		

1. Consider the matrix $A = \begin{bmatrix} 3 & 9 & 1 & -1 \\ 2 & 6 & 1 & 0 \\ 3 & 9 & 0 & -3 \end{bmatrix}$. Find a basis for the null space and for the row space of A.

$$A = \begin{bmatrix} 3 & 9 & 1 & -1 \\ 2 & 6 & 1 & 0 \\ 3 & 9 & 0 & -3 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 2 & 6 & 1 & 0 \\ 3 & 9 & 0 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 3 & 9 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1,3,0,-1),(0,0,1,2)\}$ is a basis of the row space of A.

Let us find the solution of $AX = 0$:
The solution of $x_1 + 3x_2 - x_4 = 0$ and $x_3 + 2x_4 = 0$ satisfies

$$(x_1, x_2, x_3, x_4) = (-3x_2 + x_4, x_2, -2x_4, x_4) = x_2(-3,1,0,0) + x_4(1,0,-2,1).$$

Therefore, $\{(-3,1,0,0),(1,0,-2,1)\}$ is a basis for the null space of A.

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2. Let W be the subspace of the space of polynomials on $\mathbb{R}$ spanned by the vectors $P_1 = 1 + X$, $P_2 = 1 - X + X^2$ and $P_3 = 2X^2 + X^3$. Determine whether the vector $Q = 1 + \lambda X + X^3$ is contained in W.

The vector Q is contained in W if and only if $c_1P_1 + c_2P_2 + c_3P_3 = Q$ has a solution.

$$\begin{aligned} c_1(1+X) \quad + c_2(1 - X + X^2) \quad + c_3(2X^2 + X^3) &= 1 + \lambda X + X^3 \\ \text{if and only if} \quad (c_1 + c_2) + (c_1 - c_2)X + (c_2 + 2c_3)X^2 + c_3X^3 &= 1 + \lambda X + X^3 \\ \text{if and only if} \quad c_3 = 1, c_2 = -2, c_1 = 3 &= \lambda - 2. \end{aligned}$$

Therefore, $Q \in W$ if and only if $\lambda = 5$.

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3. Let V and W be two vector spaces over a field F, let $T : V \to W$ be a linear transformation and let U be a subspace of V. Prove that $T(U) = \{T(x) : x \in U\}$ is a subspace of W.

The vector $0 \in V$ is contained in U. Hence, $T(0) = 0$ is contained in $T(U)$. Thus, $T(U)$ is not empty.

Suppose that $x, y \in T(U)$ and $k \in F$. Since There exist $x', y' \in U$ such that $T(x') = x, T(y') = y$. Since U is a subspace of V, we have $kx' + y' \in U$. Then

$$kx + y = kT(x') + T(y') = T(kx' + y')$$

is in $T(U)$. Therefore, $T(U)$ is a subspace of W.

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4. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis of $\mathbb{R}^3$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 1, 1)$ and $\alpha_3 = (0, 0, 1)$. Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x, y, z) = (y + z, x, 0)$.

(a) Find the matrix of T relative to the bases $\mathcal{B}_{\text{st}}$ and $\mathcal{B}$, where $\mathcal{B}_{\text{st}}$ is the standard basis.

Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_1 = (0, 0, 1)$.

$$\begin{aligned} T(e_1) &= T(1, 0, 0) = (0, 1, 0) = \alpha_1 + \alpha_2 - 2\alpha_3, \\ T(e_2) &= T(0, 1, 0) = (1, 0, 0) = \alpha_1 - \alpha_3, \\ T(e_3) &= T(0, 0, 1) = (1, 0, 0) = \alpha_1 - \alpha_3. \end{aligned}$$

Thus the matrix of T relative to the bases $\mathcal{B}_{\text{st}}$ and $\mathcal{B}$ is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & -1 & -1 \end{bmatrix}$.

(b) Find the matrix $[T]_{\mathcal{B}}$ relative to $\mathcal{B}$.

$$\begin{aligned} T(\alpha_1) &= T(1, 0, 1) = (1, 1, 0) = 2\alpha_1 + \alpha_2 - 3\alpha_3, \\ T(\alpha_2) &= T(-1, 1, 1) = (2, -1, 0) = \alpha_1 - \alpha_2, \\ T(\alpha_3) &= T(0, 0, 1) = (1, 0, 0) = \alpha_1 - \alpha_3. \end{aligned}$$

Thus the matrix of T relative to the bases $\mathcal{B}_{\text{st}}$ and $\mathcal{B}$ is $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ -3 & 0 & -1 \end{bmatrix}$.

(c) What are the rank and the nullity of T?

rank (T) =rank (A) = rank $[T]_{\mathcal{B}} = 2$. By the rank-nullity theorem, the nullity of T is $3-$rank $(T) = 2$.

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5. Let $S : V \to W$ and $T : W \to Z$ be linear transformations between vector spaces over the same field.

(a) Prove that $T \circ S$ is linear.

For $x, y \in V$ and $k \in F$,

$$\begin{aligned} (T \circ S)(kx + y) &= T(S(kx + y)) && S \text{ is linear} \\ &= T(kS(x) + S(y)) && T \text{ is linear} \\ &= kT(S(x)) + T(S(y))) \\ &= k(T \circ S)(x) + (T \circ S)(y)). \end{aligned}$$

(b) Prove that $\mathrm{Ker}(S) \subset \mathrm{Ker}(T \circ S)$.

Let $x \in$Ker(S), i.e. $S(x) = 0$. Since T is linear, $(T \circ S)(x) = T(S(x)) = 0$. Hence, $x \in$Ker$(T \circ S)$, proving that $\mathrm{Ker}(S) \subset \mathrm{Ker}(T \circ S)$.

(c) Prove that if the vector spaces V, W, Z are finite dimensional, then

$$\mathrm{rank}(T \circ S) \leq \min\{\mathrm{rank}(T), \mathrm{rank}(S)\}.$$

Since $S(V) \subset W$, we get $T(S(V)) \subset (W)$, i.e. Im$(T \circ S) \subset$Im(T). Hence

$$\mathrm{rank}(T \circ S) = \dim \mathrm{Im}(T \circ S) \leq \dim \mathrm{Im}(T) = \mathrm{rank}(T).$$

By part (b), $\dim \mathrm{Ker}(S) \leq \dim \mathrm{Ker}(T \circ S)$. By Rank-Nullity Theorem,

$$\mathrm{rank}(T \circ S) = \dim V - \dim \mathrm{Ker}(T \circ S) \leq \dim V - \dim \mathrm{Ker}(S) = \mathrm{rank}(S).$$

Therefore, we have

$$\mathrm{rank}(T \circ S) \leq \min\{\mathrm{rank}(T), \mathrm{rank}(S)\}.$$